Mode fluctuation distribution of coupled quartic oscillators

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The mode fluctuation distribution (MFD) of a system of two quartic oscillators coupled by a quartic perturbation is numerically studied. The coupling strength serves as a control parameter to simulate the transition from integrable to chaotic regimes. It is demonstrated that even in this potential system the MFD turns out to be a very sensitive measure, as it manifests the same transformation seen in billiard systems. It is characterized by Gaussian and skewed distributions in the regions, known to be chaotic and integrable, respectively, from classical Poincaré sections and quantum-mechanical level spacing statistics. In the intermediate regions where the Kol'mogorov-Arnol'd-Moser tori survive, the MFD has various distorted forms. [S1063-651X(98)04712-6]

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Distribution of quantum energy levels in classically chaotic systems has been studied in recent years. Nearest neighbor spacing distribution (NNSD), spectral rigidity, distribution of the values of the wave functions, level curvatures with respect to internal parameters, etc. have been shown to be useful to investigate irrespective of the systems being classically chaotic or integrable. For example, the NNSD becomes the Wigner distribution for chaotic systems, whereas it becomes the Poisson distribution if systems are integrable [1]. However a few years ago an exception was found in the hyperbolic billiard, which is always expected to be classically chaotic. In a certain subtle boundary condition, the NNSD occasionally becomes almost the Poisson distribution [2–5].

The mode fluctuation distribution (MFD) has been proposed as an alternative measure that detects the chaotic nature of systems [4,5]. It is predicted that the MFD of chaotic systems always becomes the Gaussian distribution irrespective of boundary conditions, and that the MFD of integrable systems always clearly deviates from the Gaussian. It is numerically shown that the MFD in the hyperbolic billiard is consistent with the Gaussian distribution and independent of the boundary conditions. Recently the MFD's of several integrable and chaotic billiards have been studied [6,7]. These results also support the MFD hypothesis. For example, eliminating the contribution of the bouncing ball orbits carefully, the MFD becomes very close to the Gaussian distribution [6] in the Bunimovich stadium billiard.

So far the hypothesis has been checked only on billiard systems, but it has not been tested for potential systems. In this Brief Report we choose a system of two quartic oscillators coupled by a quartic term to each other. Previously similar systems have been intensively studied employing other methods [8–11]. This system is known to be a suitable example that can be transmuted from an integrable system to a chaotic system continuously by changing a single coupling parameter. It has been investigated that the NNSD evolves from a Poisson distribution to a Wigner distribution under transmutation [8]. The spectral rigidity Δ_3 has also been studied, and shows a similar evolution [8]. In the following we show that the MFD behaves as the Gaussian distribution in the chaotic regime, a characteristic skew distribution in the integrable limit, and various other distorted distributions in the intermediate regimes.

With a bounded potential a quantum system has a discrete energy spectrum $\{E_n\}$, which defines a spectral staircase function $N(E)\equiv\Sigma_{n=1}^{\infty}\theta(E-E_n)$, and the spectral density $d(E)\equiv dN(E)/dE$. The staircase function can be separated into a mean smooth part $\langle N(E)\rangle$ and a mode fluctuating part $N_{\mathrm{fl}}(E)$: $N(E)=\langle N(E)\rangle+N_{\mathrm{fl}}(E)$. Note that the bracket $\langle\cdots\rangle$ denotes the average over an interval which is much larger than the mean energy level spacing $\langle d\rangle^{-1}$, and sufficiently smaller than the energy E under consideration.

It is expected that our system has a saturated value $\Delta_\infty(E)$ of the spectral rigidity $\Delta_3(L,E)$ at sufficiently large L, as in the billiard systems [4,5,12]. The saturated value turns out to be the second moment of the fluctuating part, i.e.,

$$\Delta_\infty(E)\Rightarrow\left\langle\frac{\langle d\rangle}{L}\int_{-[L/2\langle d\rangle]}^{[L/2\langle d\rangle]}N_{\mathrm{fl}}(E+\varepsilon)^2d\varepsilon\right\rangle\quad \text{at } L\gg L_{\max}, \tag{1}$$

where $L_{\max}$ corresponds to a scale $h\langle d\rangle/T_{\min}$, which is an energy scale $h/T_{\min}$ normalized by the mean energy level spacing $\langle d\rangle^{-1}$, and $T_{\min}$ is the period of the shortest classical closed orbit in the system. The MFD is the normalized distribution of the fluctuating part $N_{\mathrm{fl}}(E)$. It is defined as the distribution of a variable,

$$W(E)=\frac{N_{\mathrm{fl}}(E)}{\sqrt{\Delta_\infty(E)}}. \tag{2}$$

Thus its average must be zero, and its variance must be 1.

The Hamiltonian of the coupled quartic oscillators that we choose is

$$H=\tfrac{1}{2}(p_x^2+p_y^2)+3x^4+y^4-\lambda x^2y^2. \tag{3}$$

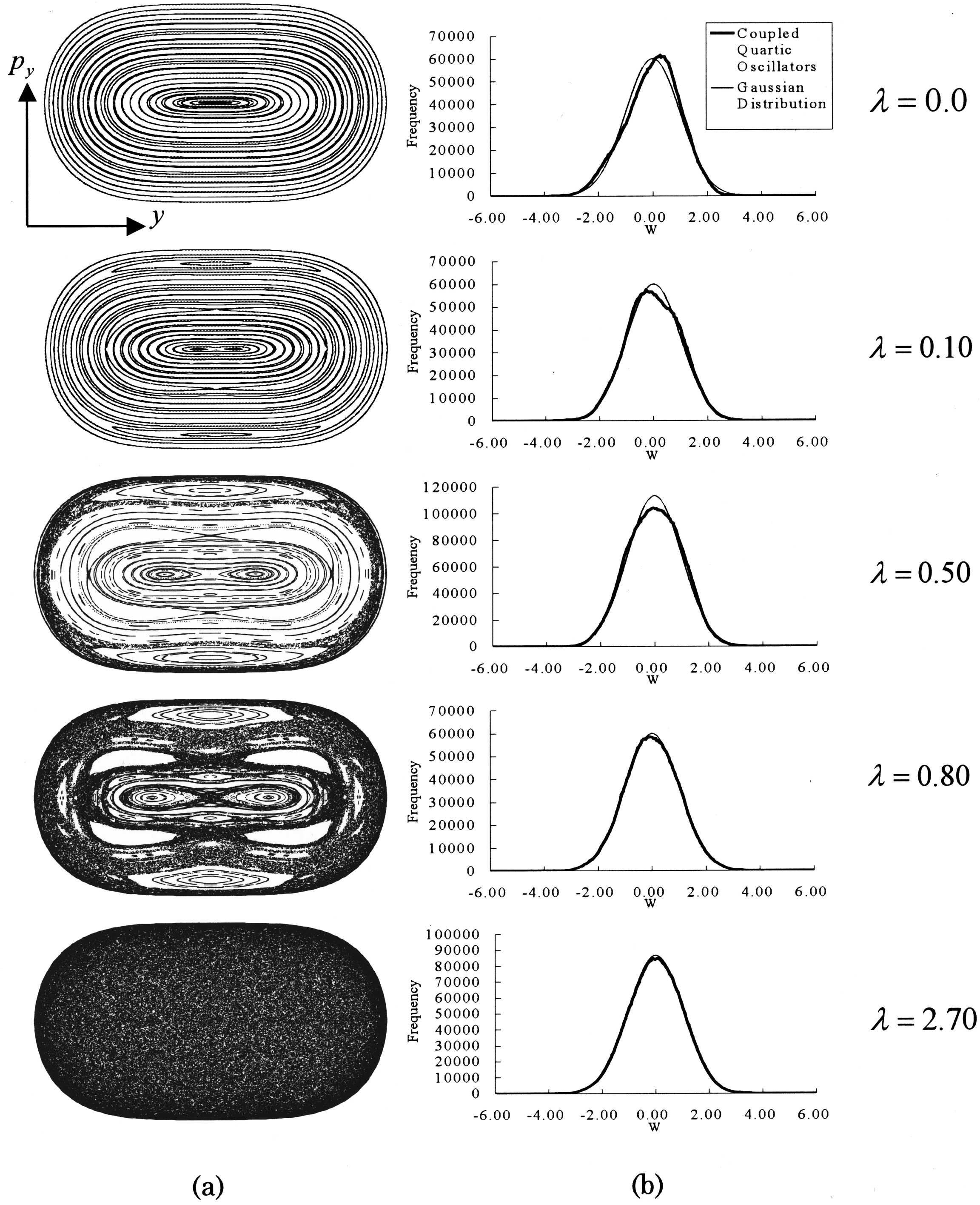

FIG. 1. (a) The Poincaré sections corresponding to the Hamiltonian given by Eq. (3): (y,p_y) section at $x=0$. (b) The mode fluctuation distributions which are given by the histogram of the variable $W(E)$ defined by Eq. (2).

Note that we set dimensionless units $\hbar=m=1$ $(h=2\pi)$, where m is the mass of a classical point particle in the system. The last term $-\lambda x^2y^2$ distorts the two-dimensional quartic potential, but keeps the potential $V(x,y)=3x^4+y^4-\lambda x^2y^2$ homogenous. For $\lambda=0$, this system becomes separable, that is, a superposition of two integrable systems with quartic potentials. If $\lambda>2\sqrt{3}$, the potential $V(x,y)$ no longer forms a bound quantum system. Then the search for a discrete spectrum of the system becomes meaningless. Therefore, we must keep $\lambda\leqslant 2\sqrt{3}$ in order to calculate the eigenvalues.

One of the most important properties of our classical system is its scaling property. The system is invariant under the transformation $x\rightarrow cx$, $y\rightarrow cy$, $t\rightarrow t/c$, $E\rightarrow c^4E$, owing to the homogenous potential $V(x,y)$. This scaling relation leads to some advantages in classical dynamics. The Poincaré surfaces of section, the equipotential surfaces, and the orbits in the phase space are geometrically similar under the scaling of energy. Their sizes are just c times larger in the directions of the coordinates, and c^2 times larger in the directions of the momenta. Thus, once we find a closed orbit at a certain energy, there always exists a corresponding closed orbit at any

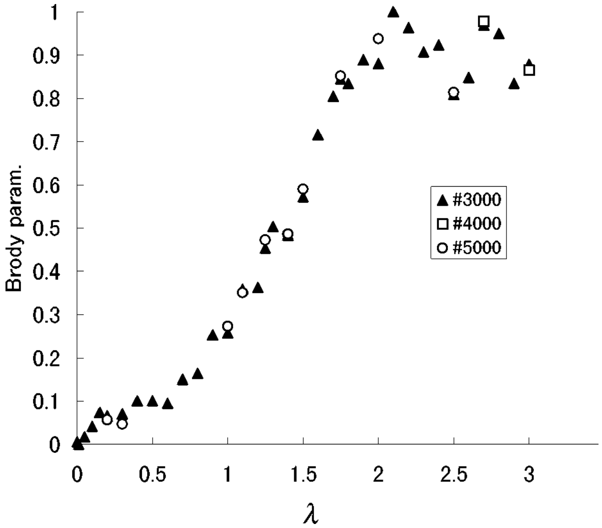

FIG. 2. Evaluated Brody parameters as a function of the coupling strength λ.

other energy. Its length ℓ and period T are transformed as $\ell \to c\ell$ and $T \to T/c$, if $E \to c^4 E$. Therefore the classical dynamical properties are geometrically similar under the scaling relations. As the simplest example, we can determine the shortest period of the unstable and isolated closed orbit $T_{\min}$ as a function of energy E. The shortest periodic orbit is always a straight line, with $y=0$ along the x axis, and we obtain $T_{\min}=\alpha E^{-1/4}$ from the scaling relation. The explicit value of α is given below.

In order to investigate the dynamical structure of our system, we numerically calculate orbits for various initial conditions and evaluate Poincaré sections. Roughly speaking, the larger λ goes, the more chaotic the system becomes [Fig. 1(a)]. If λ is closer to $2\sqrt{3}$ ($\cong 3.464$), the islands of stability become smaller and the chaotic regions dominate more in the Poincaré section. At $\lambda=2.7$ [Fig. 1(a)], at least from our numerical investigation, there remains no island in the Poincaré sections. It can be regarded as a chaotic system. The study of the Poincaré sections also shows that the last Kol'mogorov-Arnol'd-Moser (KAM) collapses somewhere very close to $\lambda=0.80$ [Fig. 1(a)].

We compute the energy levels by numerical diagonalization of the truncated matrix of Hamiltonian (3) in the basis of two independent harmonic oscillators [13]. We calculate 3000–5000 eigenvalues out of about 8500–15 000-dimensional truncated matrices of Hamiltonian (3). The convergence of the calculated eigenenergies has been confirmed by changing the frequencies of two harmonic oscillators. Only eigenenergies corresponding to wave functions of even-even parity are calculated. This means that what we actually calculate is the desymmetrized system: a quarter of the system, where a classical point particle is confined in a region of $x \geqslant 0$ and $y \geqslant 0$, and is reflected by walls at $x=0$ and $y=0$.

The Brody parameter is evaluated for each value of λ [14]. Each data point in Fig. 2 is evaluated by using 3000–5000 energy levels from the ground state. For $\lambda > 2.0$ the values of the Brody parameter vary widely in a range of about 0.2. This may be due to the precision of our calculation. The convergence of eigenvalues in a numerical diago-

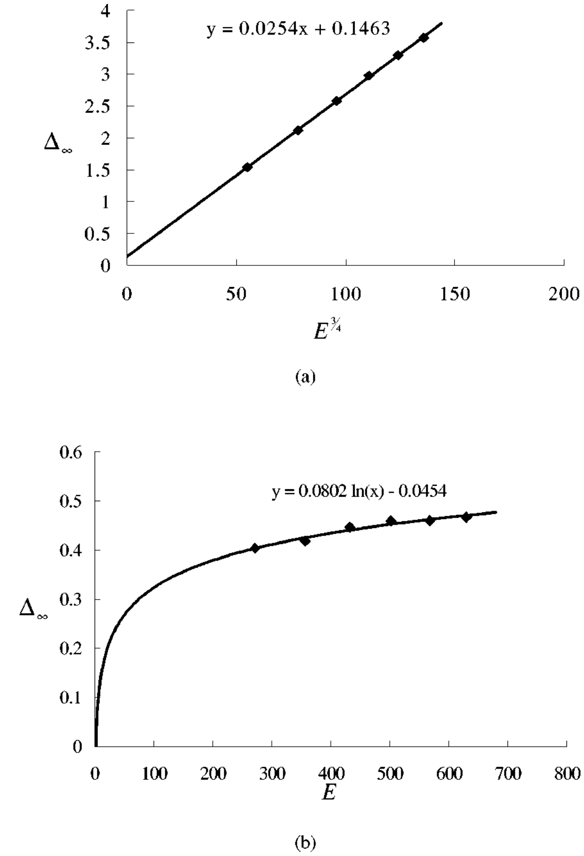

FIG. 3. The energy dependence of the saturated spectral rigidity Δ_∞ of the integrable case (a) $\lambda=0$, and the chaotic limit (b) $\lambda=2.7$. Each value of Δ_∞ represents the average value of about 60–200 sample points. It is plotted at the average energy of the sample points. The axis of abscissa for the integrable case (a) represents the $\frac{3}{4}$th power of E.

nalization of the truncated Hamiltonian becomes worse in a region of larger λ. The Brody parameter shows its plateau around $0.3<\lambda<0.8$, and after $\lambda>0.80$ it swiftly rises to 1. It corresponds remarkably to the classical mechanical feature that remaining KAM tori split the phase space into the subregions. However, after $\lambda>0.80$, an overall mixing of the phase space occurs.

For the case $\lambda=0.0$, our system becomes completely integrable. The spectral rigidity is semiclassically estimated as [12,15]

$$\Delta_\infty(E)=\frac{3}{2\pi^2}\sum_{M_1,M_2=1}^{\infty}\frac{M_1^2M_2^2/C_1^3C_2^3}{\left(\frac{M_1^4}{C_1^3}+\frac{M_2^4}{C_2^3}\right)^{7/4}}E^{3/4}\cong 0.0257E^{3/4}, \tag{4}$$

where $C_1=3^{1/3}(3\pi/2K)^{4/3}$, $C_2=(3\pi/2K)^{4/3}$ and $K=F(\pi/2,1/\sqrt{2})=1.854\,07$, with F being the complete elliptic integral of the first kind. On the other hand, from energy levels, it is numerically estimated as $\Delta_\infty(E)=0.0254E^{3/4}+0.146$ [Fig. 3(a)], and conforms to the semiclassical analysis. This implies that the Berry-Tabor treatment [15] works fairly well even in the case of a nontrivial potential system.

When the system is sufficiently chaotic, $\lambda=2.7$, we can predict the energy dependence of $\Delta_\infty(E)$ as [12]

$$\Delta_\infty(E) \cong \frac{1}{\pi^2}\ln\{eL_{\max}\} - \frac{1}{8}. \tag{5}$$

Here $L_{\max} \cong 2\pi\langle d\rangle/T_{\min}$ is the maximum value of L, above which the spectral rigidity $\Delta_3(L,E)$ is saturated, $T_{\min}$ is the shortest period of the closed orbit [12], and e is the Euler number. The scaling relations are also helpful in a quantum analysis. From the scaling relations $T_{\min} = \alpha E^{-1/4}$ and $\langle d\rangle = \beta E^{1/2}$, we have

$$\Delta_\infty(E) \cong \frac{3}{4\pi^2}\ln E + \frac{1}{\pi^2}\ln\left\{ e\frac{2\pi\beta}{\alpha}\right\} - \frac{1}{8}$$
$$= 0.0760 \ln E - 0.0616, \tag{6}$$

where $\alpha = 1.41$ and $\beta = 0.152$ are used. The value of $\alpha = (3\pi/2)C_1^{-3/4}$ is derived from an analytical calculation of the classical shortest periodic orbit. The value of β is obtained by the numerical calculation of the quantum energy density. Numerically the energy dependence of $\Delta_\infty(E)$ is fitted well as $\Delta_\infty(E) = 0.0802 \ln E - 0.0454$ for $\lambda = 2.7$ [Fig. 3(b)]. Considering the precision of our numerical calculation, it agrees well with the semiclassical prediction [Eq. (6)].

When $0.0 \leqslant \lambda < 1.0$, the MFD forms into variously distorted shapes [Fig. 1(b)], whereas the NNSD is like the Poisson distribution (Fig. 2). This complicated behavior of the MFD results from the mixture of the islands of the pseudoperiodic orbits and the chaotic regions. Particularly in the range of $0.0 \leqslant \lambda < 0.1$, the NNSD is very close to the Poisson distribution. However, the MFD already transforms to a very distorted form at $\lambda = 0.1$. This implies that the MFD is more sensitive to the effect of a small perturbation on the integrable system than the NNSD. The Brody parameter forms its plateau around $0.3 < \lambda < 0.8$ (Fig. 2), while the MFD changes rapidly and is already close to the Gaussian distribution at $\lambda = 0.8$ [Fig. 1(b)]. When $\lambda > 1.0$, the MFD already becomes very close to the Gaussian form. On the other hand, the NNSD transforms toward the Wigner distribution continuously. The Brody parameter also indicates that the NNSD becomes very close to the Wigner distribution when $\lambda > 2.0$ (Fig. 2). Especially at $\lambda = 2.7$, the Brody parameter is very close to 1, which corresponds to the Wigner distribution.

In this paper, we have examined coupled quartic oscillators whose Hamiltonian is given by Eq. (3). The Hamiltonian can simulate the transition from integrable to chaotic systems by the variation of a single parameter λ. This feature is confirmed by an analysis of the corresponding classical Poincaré sections and the quantum mechanical NNSD. In order to analyze the MFD, the spectral rigidity is calculated. The existence of saturated values of the spectral rigidity is numerically confirmed, and corresponding values in the integrable and chaotic domains conform to those deduced from the semiclassical arguments. In the regime where the system is sufficiently chaotic ($\lambda = 2.7$), the MFD becomes almost indistinguishable from the Gaussian distribution. The system can be set exactly integrable ($\lambda = 0.0$), and then the MFD expresses a characteristic asymmetry as in the square billiard [16], and as in stadium billiards with bouncing ball modes [6,7]. Although it presents a Gaussian distribution for the chaotic limit, the MFD is more likely to manifest the skewness that is characteristic of the integrable system. It cannot tell the difference between hard chaos and soft chaos after the last KAM has disintegrated. Therefore, we conclude that it should be the measure of the integrability rather than the chaoticity not only in billiard systems but also in potential systems, as seen from the example of coupled oscillators described here.

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